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Raiders of the lost AdS

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Abstract: We demonstrate that under certain conditions a theory of conformal quantum mechanics will exhibit the symmetries of two half-Virasoro algebras. We further demonstrate the conditions under which these algebras combine to form a single-Virasoro algebra, and comment on the connection between this result and the AdS_2/CFT_1 correspondence.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry.

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1. Introduction

Indy: I'm going after that [AdS].
Sallah: How?
Indy: I don't know.
I'm making this up as I go.
"Raiders of the Lost Ark"

Recently, there has been great interest in the AdS/CFT correspondence ([1]–[3], and many others). Although there has been a great deal of work related to the study of AdS_3, AdS_4, AdS_5 and AdS_7, there has been relatively little study of the AdS_2 case. Type-II supergravity exhibits solutions of the form $AdS_2 \times S^2 \times \mathbb{T}^6$, $AdS_2 \times S^3 \times \mathbb{T}^5$ and other solutions related to Calabi-Yau compactification.

On one hand, the AdS_2/CFT_1 correspondence has the potential to be extremely fascinating. A weakly curved $AdS_2 \times S^2$ space will be an approximately flat 4-dimensional space. As a result, the AdS_2/CFT_1 correspondence naively may be able to make non-trivial statements about 4-dimensional quantum gravity. On the other hand, there are potential obstacles to a true understanding or complete formulation of the AdS_2/CFT_1 correspondence, such as the fragmentation of AdS_2, etc.

Nevertheless, a clearer picture of the relationship between AdS_2 and CFT_1 has recently emerged [4]–[21]. In particular, it was noted in [4] that any theory of quantum gravity in the background of AdS_2 contains not only an $\mathrm{SL}(2,\mathbb{R})$ symmetry, but also the symmetries of a Virasoro algebra. This may imply that any boundary theory holographically dual to AdS_2 should also contain the symmetries of a Virasoro algebra. In [14], it was shown that any classical scale-invariant mechanics of one variable exhibited not only conformal invariance, but also the symmetries of a Virasoro algebra. In a few instances, this result was extended to a quantum mechanical statement of the commutation relations between generators (as opposed to a statment regarding Poisson bracket relations between generator functions). However, a general statement regarding multi-particle systems and operator algebras was still missing.

In this paper we demonstrate that, under certain conditions, a theory of conformal quantum mechanics (of an arbitrary number of variables) exhibits the symmetries of two half-Virasoro algebras. If a further condition holds, then these two algebras can be combined to form a single-Virasoro algebra.

1.1 Classical generators

The conformal group in $0+1$ dimensions is $\mathrm{SL}(2,\mathbb{R})$. The algebra can be written as

$$[D,H] = \imath\hbar H\,, \qquad [D,K] = -\imath\hbar K\,, \qquad [H,K] = 2\imath\hbar D\,. \tag{1.1}$$

If one makes the identification

$$L_{-1} = -\frac{\imath}{\hbar}\,H\,, \qquad L_0 = -\frac{\imath}{\hbar}\,D\,, \qquad L_1 = \frac{\imath}{\hbar}\,K\,, \tag{1.2}$$

then the conformal algebra is identical to the $\mathrm{SL}(2,\mathbb{R})$ algebra formed by the global subalgebra of the Virasoro algebra (note that there are other ways of embedding the conformal algebra in the Virasoro algebra).

In [14] it was shown that if the conformal algebra of a theory of classical conformal mechanics is embedded in the global subalgebra of the Virasoro algebra as shown above, then generators of a full-Virasoro algebra can be found.

In the examples discussed in [14], one can see that the Virasoro generators can be written in the form[1]

$$L_m = L_0^{1+m} L_{-1}^{-m}\,. \tag{1.3}$$

One can easily show at the level of Poisson brackets that if $\{L_0, L_{-1}\}_{PB} = L_{-1}$, then the generators defined above satisfy the algebra

$$\begin{aligned}\{L_m, L_n\}_{PB} &= (1+m)(-n)\, L_0^{1+m+n} L_{-1}^{-m-n-1}\{L_0, L_{-1}\}_{PB} + \\ &\quad + (1+n)(-m)\, L_0^{1+m+n} L_{-1}^{-m-n-1}\{L_{-1}, L_0\}_{PB} = \\ &= \Big[(1+m)(-n) - (1+n)(-m)\Big]\, L_{m+n} = \\ &= (m-n)\, L_{m+n}\,. \end{aligned} \tag{1.4}$$

[1]We thank Sangmin Lee for making this point.

Thus, any system of classical mechanics with scale-invariance also exhibits the symmetries of a full-Virasoro algebra (at the level of Poisson brackets). This is a generalization of the work done in [14], as it applies to the case of an arbitrary number of variables. However, the analysis is still only classical in nature. In order to make a statment about conformal quantum mechanics, one must also understand the issues related to the normal ordering of operators.

1.2 Quantum generators

Suppose that the generators L_0, L_1 and L_{-1} satisfy the $\mathrm{SL}(2,\mathbb{R})$ algebra given by

$$[L_0, L_1] = -L_1\,, \qquad [L_0, L_{-1}] = L_{-1}\,, \qquad [L_1, L_{-1}] = 2L_0\,. \tag{1.5}$$

If the operator L_{-1} is invertible, then we may consistently define the operator L_{-1}^{-1}. This operator has the commutation relation $[L_0, L_{-1}^{-1}] = -L_{-1}^{-1}$. We then make the ansatz

$$L_m = L_0\left(L_{-1}^{-1}L_0\right)^m = \left(L_0 L_{-1}^{-1}\right)^m L_0\,, \quad m \geq 0\,. \tag{1.6}$$

One can easily see that

$$\begin{aligned}
[L_m, L_n] &= L_0\left(L_{-1}^{-1}L_0\right)^m L_0\left(L_{-1}^{-1}L_0\right)^n - L_0\left(L_{-1}^{-1}L_0\right)^n L_0\left(L_{-1}^{-1}L_0\right)^m = \\
&= (mL_{m+n} + L_0L_{m+n}) - (nL_{m+n} + L_0L_{m+n}) = \\
&= (m-n)L_{m+n}\,,
\end{aligned} \tag{1.7}$$

where m,$n \geq 0$. Similarly, one finds that

$$\begin{aligned}
[L_m, L_{-1}] &= L_0\left(L_{-1}^{-1}L_0\right)^m L_{-1} - L_{-1}L_0\left(L_{-1}^{-1}L_0\right)^m = \\
&= L_0\left(L_{-1}^{-1}L_0\right)^{m-1}(1+L_0) - L_0^2\left(L_{-1}^{-1}L_0\right)^{m-1} + L_0\left(L_{-1}^{-1}L_0\right)^{m-1} = \\
&= 2L_{m-1} + (m-1)L_{m-1} = (m+1)L_{m-1}\,.
\end{aligned} \tag{1.8}$$

It is thus clear that, given a system of conformal mechanics with $\mathrm{SL}(2,\mathbb{R})$ symmetry, the invertibility of L_{-1} implies the existence of a half-Virasoro algebra consisting of all Virasoro modes L_m with $m \geq -1$. In particular, a scale-invariant theory will also exhibit conformal invariance (with $K = -DH^{-1}D$), provided that there are no zero-energy states. This can be seen by defining $L_{-1} = -\frac{\imath}{\hbar}H$ and $L_0 = -\frac{\imath}{\hbar}D$ and applying the above results. Since H is hermitean, L_{-1} will be invertible if H has no eigenvalues which are zero. From the above construction one will find a half-Virasoro algebra with $L_1 = \frac{\imath}{\hbar}K$.

It is not entirely clear how this statement is reconciled with [16], where it was shown that a scale-invariant hamiltonian (with only quadratic momentum dependence) exhibits conformal invariance only if it admits a closed homothety. However, [16] studied the case where the special conformal symmetry generator K depended only on the position operators X, and not on their momentum conjugates. In the construction given above, however, it is clear that K will generically depend on momentum.

In an exactly analogous manner, one can consider the situation where L_1 is invertible (such that one can consistently define L_1^{-1}). In this case, we make the ansatz

$$L_m = L_0 \left(L_1^{-1} L_0\right)^{-m} = \left(L_0 L_1^{-1}\right)^{-m} L_0 \,, \quad m \leq 0 \,. \tag{1.9}$$

One finds that these operators yield a half-Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} \tag{1.10}$$

which closes for $m \leq 1$.

We might then ask whether it is possible for these two half-Virasoro algebras to be united into a single full-Virasoro algebra. The key to this question is the overlap of these algebras, namely the generators L_1, L_0 and L_{-1}. We will demand that these generators are the same in both half-Virasoro algebras. This implies the conditions

$$L_0 L_{-1}^{-1} L_0 = L_1 \,, \qquad L_0 L_1^{-1} L_0 = L_{-1} \,. \tag{1.11}$$

If L_0 is also invertible, then these two conditions are actually identical.

When (1.11) is satisfied, we can actually find a full-Virasoro algebra whose quantum generators are given by

$$\begin{aligned} L_m &= L_0 \left(L_{-1}^{-1} L_0\right)^{m} = \left(L_0 L_{-1}^{-1}\right)^{m} L_0 \,, \quad m \geq 0 \,, \\ L_m &= L_0 \left(L_1^{-1} L_0\right)^{-m} = (L_0 L_1^{-1})^{-m} L_0 \,, \quad m \leq 0 \,. \end{aligned} \tag{1.12}$$

One would like to show that the generators defined in this way satisfy the full-quantum Virasoro algebra $[L_m, L_n] = (m-n) L_{m+n} + f(m)\, \delta_{mn}$. For the case where either m,$n \geq 0$ or m,$n \leq 0$, the algebra is obviously satisfied. Consider the case $m > 0$, $n < 0$, $m + n > 0$.

$$\begin{aligned} [L_n, L_m] &= \left[\left(L_0 L_1^{-1}\right)^{-n} L_0, L_0 \left(L_{-1}^{-1} L_0\right)^{m}\right] \\ &= \left(L_0 L_1^{-1}\right)^{-n-1} L_{-1} L_0 \left(L_{-1}^{-1} L_0\right)^{m} - \left(L_0 L_{-1}^{-1}\right)^{m} L_0 L_{-1} \left(L_1^{-1} L_0\right)^{-n-1} \\ &= L_{n+1} L_{m-1} - L_{m-1} L_{n+1} - \left(L_0 L_1^{-1}\right)^{-n-1} L_{m-1} - L_{m-1} \left(L_1^{-1} L_0\right)^{-n-1} \\ &= [L_{n+1}, L_{m-1}] - 2 L_{m+n} \,. \end{aligned} \tag{1.13}$$

After anchoring the recursion relation with $[L_{-1}, L_m] = -(m+1) L_{m-1}$, one finds that the Virasoro algebra is satisfied for all m and n in the range of interest. A similar argument shows that this is also true for $m + n < 0$. Thus, one sees that any theory of conformal quantum mechanics which satisfies the above conditions also has the symmetries of a full-Virasoro algebra. In addition, it is clear from the above calculation that the algebra has no central charge.

Given a theory with operators L_0 and L_{-1} which satisfy the appropriate commutation relations, where L_{-1} is invertible, one can simply define $L_1 = L_0 L_{-1}^{-1} L_0$.

The invertiblity of L_1 implies the existence of two half-Virasoro algebras, and the invertibilty of L_0 would further imply that these two algebras combine to form a single-Virasoro algebra.

Note that, under the (1.2), $L_{-1,0,1}$ are antihermitean. It is clear that if this is the case, then the L_m's defined by (1.12) are all antihermitean. However, in the context of $(1+1)$-conformal field theory one usually defines L_m's which satisfy the hermiticity property $L_n^\dagger = L_{-n}$. If this property is satisfied by $L_{-1,0,1}$, then the L_m's defined by (1.12) satisfy it as well.

1.2.1 A simple example

Consider the hamiltonian of a non-relativistic free particle in one dimension, $H = \frac{1}{2}p^2$ (where the coordinate has been rescaled in order to absorb the mass into the conjugate momentum). If one writes the standard dilatation operator $D = \frac{1}{4}(rp+pr)$, one finds that H and D satisfy the standard commutation relation

$$[D, H] = \imath\hbar H\,. \tag{1.14}$$

We may project out of the Hilbert space all states whose wavefunctions are even under $r \to -r$ (this is, in fact, exactly what we would do if we treated this as the hamiltonian for the radial wavefunction of a free particle in three dimensions with no angular momentum). The remaining energy eigenstates have wavefunctions of the form $\psi(r) = A\sin(kr)$ with energies $E = \hbar^2 k^2/2m$. One finds that there is a continuum of eigenstates with arbitrary positive energy (as scale-invariance demands), but there is no normalizable zero-energy state (as the wavefunction for such a state would vanish everywhere). Therefore, one may invert the hamiltonian and define the operator

$$K = -DH^{-1}D = -\frac{1}{2}\,r^2 - \frac{3}{8}\hbar^2\,\frac{1}{p^2}\,. \tag{1.15}$$

This operator differs from the usual special conformal symmetry generator ($K = -\frac{1}{2}r^2$), but nevertheless is well defined and allows the conformal algebra to close. By writing K as a momentum-space operator (substituting $r = \imath\hbar\,\frac{\partial}{\partial p}$), one can easily show that K also has no normalizable eigenstates with zero eigenvalue. Using the embedding defined in (1.2), one can verify straightforwardly that the conditions (1.11) are satisfied. This means that one can write the quantum generators of a full-Virasoro algebra (1.12).

In many constructions of conformal quantum mechanics it is common for the operator $H' = \frac{1}{2}(H-K)$ (in our conventions) to be used as the hamiltonian, due to the fact that it has a discrete spectrum. In the example given above, one finds

$$H' = \frac{1}{2}(H-K) = \frac{1}{4}\,p^2 + \frac{1}{4}\,r^2 + \frac{3}{16}\,\frac{\hbar^2}{p^2}\,. \tag{1.16}$$

Under the phase space rotation $\tilde{r} = p$, $\tilde{p} = -r$, we see that this is the hamiltonian for a bound particle (indeed, it is a hamiltonian of the Calogero-Moser form), and thus has a discrete spectrum which is bounded from below.

2. Relation to AdS_2

Perhaps there is some vital bit of evidence which eludes us.
Belloq, "Raiders of the Lost Ark"

2.1 Calogero models

The naive AdS/CFT correspondence suggests that there is a boundary conformal quantum mechanics which is dual to quantum gravity on the space $AdS_2 \times S^2 \times \mathbb{T}^6$. In [7], it was suggested that this bosonic part of the boundary conformal quantum mechanics is actually given by the N particle Calogero model with hamiltonian

$$H = \frac{1}{2}\sum_{i=0}^{N} p_i^2 + \sum_{i<j} \frac{\lambda}{|r_j - r_i|^2}\,. \tag{2.1}$$

It is well known that this system has conformal symmetry. It is also well known that the hamiltonian for this system has no ground state. The results above thus indicate that the N particle Calogero Model respects the symmetries of a half-Virasoro algebra. To determine if the other half of the Virasoro algebra is also present, one would have to calculate $L_1 = L_0 L_{-1}^{-1} L_0$ and determine its spectrum, a more complicated task which will not be attempted in this work.

The Calogero model itself is a limit of the more general Calogero-Moser model, whose hamiltonian is given by

$$H = \frac{1}{2}\sum_{i=0}^{N} p_i^2 + \sum_{i<j} \frac{\lambda}{|r_j - r_i|^2} + \frac{k}{2}\sum_{i=0}^{N} r_i^2\,. \tag{2.2}$$

It is already known [8, 9, 10, 11] that the Calogero-Moser model exhibits the symmetries of a Virasoro algebra, with the embedding $L_0 = H$. Note that this is not the embedding which we discussed earlier. In fact, the Calogero-Moser model is not conformally invariant, as its hamiltonian does not have a continuous spectrum. But in the limit $k \to 0$, the Calogero-Moser model reduces to the Calogero model in question. However, in that limit generators of the Virasoro algebra used in [11] become infinite (as they contain negative powers of the constant k). This is not unexpected (when H is related to L_0) because the Calogero-Moser model has a discrete spectrum, whereas the Calogero model has a continuous spectrum.

However, in the construction given by (1.2) and (1.12), the generators of the half-Virasoro algebra are well defined. If the Calogero model is indeed related to the

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boundary conformal theory which is dual to quantum gravity on $AdS_2 \times S^2 \times \mathbb{T}^6$, then it is interesting to note that it contains at least the symmetries of a half-Virasoro algebra, whereas it is known that quantum gravity on AdS_2 contains the symmetries of a Virasoro algebra.

2.2 Probing with a test particle

One may also consider the quantum mechanics describing a test-particle in the background of AdS_2. The hamiltonian for such a particle was discussed in [12], where it was found that in the non-relativisitic near-horizon limit the bosonic part of the hamiltonian reduced to that of DFF conformal quantum mechanics [22],

$$H = \frac{p^2}{2} + \frac{2J^2}{r^2}\,. \tag{2.3}$$

It is clear that this hamiltonian has no normalizable zero-energy eigenstates, thus indicating that it respects the symmetries of a half-Virasoro algebra. Determining the invertibility of L_1 (or equivalently, K) is more complicated, and will not be attempted here. However, in the case where $J = 0$, the system simply reduces to the free particle example discussed earlier, where it is clear that K is invertible, and thus that the other half of the Virasoro algebra can also be constructed. One should note that the form of the quantum generators found here is identical to the form of the classical generators of the Virasoro algebra found in [14] for the same system (up to the terms related to normal ordering). We see now that these generators not only generate a classical symmetry under Poisson brackets, but also generate a full quantum symmetry under commutators.

3. Discussion and further research

Eaton: *We have top men working on it right now.*
Indy: *Who?*
Eaton: *... top men.*

"Raiders of the Lost Ark"

We have shown that a scale-invariant quantum mechanical system with no zero-energy states exhibits not only 0+1 conformal invariance, but also the symmetries of a half-Virasoro algebra (defined as the algebra L_m for $m \geq -1$). If the operator L_1 defined by this half-Virasoro algebra is also invertible, then the system also exhibits the symmetries of another half-Virasoro algebra (given by the generators L_m with $m \leq 1$). If these two half-Virasoro algebras have identical generators L_0, L_{-1} and L_1, then the two half-Virasoro algebras in fact form a single-Virasoro algebra with no central charge.

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It is noteworthy that these constructions of the Virasoro algebra exhibit zero central charge. From the AdS_2/CFT_1 correspondence, one would expect superconformal quantum mechanics to be the boundary theory dual to string theory on $AdS_2 \times S^2 \times \mathbb{T}^6$. The gravity theory on AdS_2 has the symmetries of a Virasoro algebra, as it is a 2D theory of gravity on a strip. This Virasoro algebra should not have a central charge when the effects of ghosts are also included. It had been speculated that this might imply that the dual superconformal quantum mechanics has the symmetries of a Virasoro algebra with no central charge. But in [19], it was shown that the construction of the Virasoro algebra given in [14] was actually contained in a larger w_∞ algebra. Calogero models have also been shown to exhibit the symmetries of a w_∞ algebra; the properties of these systems were studied extensively in [8, 9, 10]. For the case of a particle in the background of AdS_2, it was shown in [19] that the central charge associated with the Virasoro algebra of coordinate diffeomorphisms of AdS_2 is replicated by the unique central extension of the w_∞ algebra. However, our result seems to be more general because it holds for conformal theories which are not necessarily connected to AdS_2. But it seems not unreasonable to expect that a fuller examination of theories of conformal quantum mechanics of an arbitrary number of variables will also show that the Virasoro algebras found here can be extended to w_∞ algebras. If so, perhaps a study of the central extensions of this w_∞ algebra (and a comparison of this with the central charge of the diffeomorphism algebra of AdS_2) will shed more light on the connection between AdS_2 and conformal quantum mechanics.

There are several directions in which further work can proceed. It would be very interesting to understand the circumstances under which (1.11) held more systematically. So far, work has focused only on the bosonic theory; further investigation of supersymmetric generalizations of these constructions is required. It is also important to better understand the physical significance of the half-Virasoro algebras which have been found here. Finally, the connections found here between conformal quantum mechanics and the Virasoro algebra further strengthen the notion that there is a deep connection between conformal quantum mechanics and $(1+1)$-conformal field theory [4]. The relationship between this connection and the AdS/CFT correspondence should be the subject of future work.

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